IFUM 544/FT
ROM2F/96/56

NON LOCAL OBSERVABLES AND CONFINEMENT IN BF FORMULATION OF YANG-MILLS THEORY*

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Abstract

The *vev*'s of the magnetic order-disorder operators in QCD are found with an explicit calculation using the first order formulation of Yang-Mills theory.

INTRODUCTION

The definition of non local observables plays a fundamental role in the study of YM theory and QCD vacuum.Indeed the well known Wilson line operator gives one of the most widely used confinement criteria, namely the area law behaviour of its *vev*, associated with a linearly rising confining potential between static probe charges in the QCD vacuum.

The structure of the confining vacuum has been described by means of the condensation of "magnetically" charged degrees of freedom [1, 2, 3, 28] giving rise to the so-called dual superconductor model. Several hypothesis on the nature of these configurations and on the dynamical mechanism which leads to their condensation have been formulated, and even if this picture in not yet uniquely determined they are nevertheless believed to play a major role in the phenomenon of confinement.

Actually this structure of the QCD vacuum admits different possible phases [5], not all confining, which can be labeled by means of the *vev*'s of the Wilson loop W and of an other non local operator M, the 't Hooft magnetic disorder parameter. W and M give rise to the well known 't Hooft algebra [5], derived by the implicit definition of M as producing center valued singular gauge transformations along the magnetic vortex lines.

The picture of the superconductor model for QCD is analytically realized in the first order formulation of YM theory, where an explicit definition of the color magnetic operator M is easily given and the calculation of the *vev* of both M and W can be performed, displaying the expected behaviour for the confining phase.

* Contribution to CARGESE SUMMER SCHOOL, july 96

The first order form of pure Yang-Mills is described by the action functional [7, 8, 9]

$$S_{BF-YM} = \int \mathrm{Tr}(iB \wedge F + \frac{g^2}{4} B \wedge *B) \tag{1.1}$$

where $F \equiv \frac{1}{2} F^a_{\mu\nu} T^a dx^\mu \wedge dx^\nu \equiv dA + i[A, A]$, $D \equiv d + i[A, \cdot]$ and B is a Lie valued 2-form. The generators of the $SU(N)$ Lie algebra in the fundamental representation are normalized as $\mathrm{Tr} T^a T^b = \delta_{ab}/2$ and the $*$ product (Hodge duality) for a p form in d dimension is defined as $* = \epsilon^{i_1 \dots i_d}/(d-p)!$. The classical gauge invariance of (1.1) is given by $\delta A = D\Lambda_0$, $\delta B = -i[\Lambda_0, B]$ and the standard YM action is recovered performing path integration over B or using the field equations

$$F = \frac{ig^2}{2} * B \qquad , \qquad DB = 0 \quad . \tag{1.2}$$

Off shell B does not satisfy a Bianchi identity and this fact should be related with the introduction in the theory of magnetic vortex lines. We remark that the short distance quantum behaviour of (1.1) is the same as in standard YM as it has been explicitely checked [12].

The action functional defined by the first term in the r.h.s. of (1.1) is known as the 4D pure bosonic BF-theory and defines a topological quantum field theory [11]. Then the bosonic YM theory can be viewed as a *perturbative expansion* in the coupling g around the topological pure BF theory, the explicit topological symmetry breaking term $\sim g^2 B^2$ in (1.1) introducing local degrees of freedom in the topological theory.

The presence of the Lie-algebra valued two-form B field in (1.1) allows the definition of an observable gauge invariant operator

$$M(\Sigma, C) \equiv \mathrm{Tr} \exp\{ik \int_\Sigma d^2y \; \mathrm{Hol}^y_{\bar{x}}(\gamma) B(y) \mathrm{Hol}^{\bar{x}}_y(\gamma')\} \quad , \tag{1.3}$$

where $\mathrm{Hol}^y_{\bar{x}}(\gamma)$ denotes the usual holonomy along the open path $\gamma \equiv \gamma_{\bar{x}y}$ with initial and final point $\bar{x}$ and y respectively, $\mathrm{Hol}^y_{\bar{x}}(\gamma) \equiv P\exp(i \int_{\bar{x}}^y dx^\mu A_\mu(x))$, where k is an arbitrary expansion parameter, $\bar{x}$ is a *fixed* point over the orientable surface $\Sigma \in M^4$ and the relation between the assigned paths γ, γ' over Σ and the closed contour C is the following: C starts from the fixed point $\bar{x}$, connects a point $y \in \Sigma$ by the open path $\gamma_{\bar{x}y}$ and then returns back to the neighborhood of $\bar{x}$ by $\gamma'_{y\bar{x}}$, (which is not restricted to coincide with the inverse $(\gamma_{\bar{x}y})^{-1} = \gamma_{y\bar{x}}$). From the neighborhood of $\bar{x}$ the path starts again to connect another point $y' \in \Sigma$. Then it returns back to the neighborhood of $\bar{x}$ and so on until all points on Σ are connected. The path $C_{\bar{x}} = \{\gamma \cup \gamma'\}$ is generic and we do not require any particular ordering prescription as it is done in similar constructions devoted to obtain a non abelian Stokes theorem [13]. Of course the quantity (1.3) is path dependent and our strategy is to regard it as a loop variable once the surface Σ is given.

Using the hamiltonian formalism it is possible to show [7] that $M(C)$ generates a local singular (or equivalently a multivalued regular) gauge transformation, $\Omega_C(\vec{x})$, along C: this is precisely the defining property of the ‘t Hooft color magnetic variable [5]. Using the classical constraints which arise from the action (1.1) it is possible to generate the classical gauge transformations; in order to have first class constraints the field content of the theory has to be enlarged, including a Lie valued vector field η. This field corresponds to “topological” degrees of freedom which in our case become dynamical [14].

Given the classical algebra of tranformations, when switching to operator valued quantities, and considering the ordering procedures required by quantization,, one obtains (assuming $\Sigma \sim S^2$ and k small)

$$M(C)|A>\simeq \mathrm{Tr}\{1\!\!1 + 2ik\oint_C dy^i \int_{\Sigma\sim S^2} d\sigma^{rs}_{(x)}\epsilon_{irs}\delta^{(3)}(\vec{y}-\vec{x})\delta^{ab}T_aT_b\}|A> \quad . \tag{1.4}$$

The integral in (1.4) can be expressed as

$$\begin{aligned} &\oint_C dy^i \int_\Sigma d\sigma^{rs}_{(x)}\epsilon_{irs}\delta^{(3)}(\vec{y}-\vec{x}) = -\oint_C dy^i \int_\Sigma \frac{d\sigma^{rs}_{(x)}}{4\pi}\epsilon_{irs}\frac{\partial}{\partial x^l}\Big[\frac{\partial}{\partial x_l}\Big(\frac{1}{|\vec{y}-\vec{x}|}\Big)\Big] \\ \simeq\ & \lim_{\epsilon\to 0}\frac{1}{4\pi}\oint_C dy^i \oint_{C'} dx^r \epsilon_{irl}\frac{\partial}{\partial x_l}\Big(\frac{1}{|\vec{y}-\vec{x}|}\Big) = \mathrm{Link}(C,C')_{\epsilon\to 0} \equiv \mathrm{sLink}(C) \quad , \end{aligned} \tag{1.5}$$

where ϵ gives a point splitting regularization between C and C', where $\Sigma = \Sigma' \cup \Sigma''$, $\partial\Sigma' = C'$ and C' encloses the singularity of (1.5). The above linking number is the so-called self-linking number of C [15]. While the linking number of two separated loops is not an invariant quantity in 4D, the quantity in (1.5) is well defined and finite, takes integer values and equals the number of windings of C' around C.

Putting the above formulae in (1.4), we find

$$M(C)|A>\simeq \mathrm{Tr}\{1\!\!1(1+2ikc_2(t)\mathrm{sLink}(C))\}|A> \quad , \tag{1.6}$$

where $\delta^{ab}T_aT_b = c_2(t)1\!\!1$. Eq. (1.6) implies that $M(C)$ generates an infinitesimal multi-valued gauge transformation; whenever C' winds $n \equiv \mathrm{sLink}(C)$ times around C, $M(C)$ creates a magnetic flux [5]

$$\Phi_C \equiv \frac{2kc_2(t)}{g}\mathrm{sLink}(C) \quad . \tag{1.7}$$

The finite multivalued gauge transformation $\Omega_C[\vec{x}]$ generated by the action of $M(C)$ over some state functional is given by

$$M(C)|A(\vec{x})>= |\Omega_C^{-1}[\vec{x}](A(\vec{x})+id_{\vec{x}})\Omega_C[\vec{x}]>\simeq \mathrm{Tr}\{e^{ig\Phi_C}1\!\!1\}|A(\vec{x})> \quad . \tag{1.8}$$

Because of the multi-valued nature of $\Omega_C[\vec{x}]$, since $A^{\Omega_C} \equiv \Omega_C^{-1}(A+d)\Omega_C$ should always be single valued, Ω_C must be in the center of $SU(N)$. To recover the standard form of the center, we normalize the free expansion parameter as $k = 2\pi/Nc_2(t)$. With these normalizations the form of the color magnetic flux is given by $\Phi_C = 2\pi n/Ng$ and the ‘t Hooft algebra is easily recovered [5].

COMPUTATION OF $< M(\Sigma, C) >$

In this section we compute the average of the BF-observable $M(\Sigma, C)$ and precisely we consider the normalized connected expectation value $< M(C) >_{conn} \equiv \frac{<M(C)>}{<1>}$.

In order to perform calculations we assume the scheme of the abelian projection gauge, in which $SU(N)$ is partially gauge fixed to an abelian subgroup [6]. In general this should be implemented using an interpolating gauge [16]. Choosing (for example) $Y_{ij} = (F_{12})_{ij}/(\lambda_i - \lambda_j)$, $\quad i \neq j = 1, \ldots, N$, where the λ's are the eigenvalues of F, the proposed gauge condition is $Y^{ch} + \xi D^0 * A^{ch} = 0$ where the superscripts $ch, 0$ stand for

the off-diagonal and diagonal part of the matrix $A = A^0 + A^{ch}$ and D^0 is the covariant derivative with respect to the diagonal part of the gauge field. Interpolating gauges are such that for short distances the relevant gauge is $D^0 * A^{ch} = 0$ and the theory is renormalizable, while for large distances the gauge is $Y^{ch} = 0$. In standard YM theory we can find in the adjoint representation only a composite of the $F_{\mu\nu}$ (and its covariant derivatives), but the dependence of $F_{\mu\nu}$ and $D^0 * A^{ch}$ from the momentum p_μ is the same. Therefore the dominance argument doesn't apply. This is why the field Y_{ij} was introduced. Unfortunately this fact makes this gauge difficult to implement using standard quantum YM fields.

Quite remarkably, these problems are not present in the BF-YM theory due to the presence of the microscopic B field. The interpolating gauge can now be easily implemented choosing, for instance, $B^{ch}_{12} + D^0 * A^{ch} = 0$. Equivalently in our formalism we can diagonalize the two-form B on the surface Σ, $\tilde{B} = V^{-1}BV = diag(\beta_1, ..., \beta_N)$, $\beta_1 \geq \beta_2 \geq \ldots \geq \beta_N$, with V being the singular gauge transformation needed to implement the abelian gauge, and then use the background gauge condition $D^0 * A^{ch} = 0$ in the renormalization program. Indeed this is the way in which magnetic configurations, related to the abelian unbroken group, should enter the theory in the large distance regime and fill the vacuum, while in the same limit we may neglect the massive off-diagonal degrees of freedom B^{ch}, A^{ch}. This approximation is often called Abelian dominance [6, 10].

The existence of monopoles in the Abelian projection gauge is due to the compactness of the $U(1)^{N-1}$ group and it is related to the existence of non trivial topological objects for the entire $SU(N)$ theory. The reducibility of the $SU(2)$ gauge connection, implies that the gauge bundle is split and thus requires the existence of a positive definite first Pontrjagin class and intersection number (which we will define later on). This fact, in its turn, implies the absence of anti-self-dual harmonic (closed) two forms [17].

In the general $SU(N)$ case we now rewrite the functional integral in terms of the variables $\alpha_i = [A^0]_{ii}$, $i = 1, ..., N$, $\sum_i \alpha_i = 0$ and $f_i \equiv [F^0]_{ii} = d[A^0]_{ii}$. It is convenient to rescale the previous geometrical fields A^0, B^0 to the physical ones $A^0 \to gA^0$, $B^0 \to \frac{1}{g}B^0$. Furthermore, we replace the surface integral in the definition of M with the integration over the so-called Poincarè dual form, ω_Σ, of (the homology class of) the surface Σ [8]. By definition, ω_Σ is closed. Moreover, choosing an orientation of the four manifold and remembering the absence of anti-self-dual harmonic two forms (imposed by requiring the existence of a reducible gauge connection) ω_Σ is chosen to be self-dual *i.e.* $*\omega_\Sigma = \omega_\Sigma$ and with the property that (up to gauge one-forms) for a generic two-form t

$$\int_\Sigma t \simeq \int_{M^4} \omega_\Sigma \wedge t \quad . \tag{2.1}$$

In a local system of coordinates (x, y, u, v) on Σ, so that Σ is given by the equations $x = y = 0$, the dual form ω_Σ can be taken as $\omega_\Sigma \simeq \delta^{(2)}(x, y)dx \wedge dy$ and normalized as

$$\int_{N(\Sigma)} \omega_\Sigma \simeq \int \delta^{(2)}(x, y)dx \wedge dy = 1 \quad , \tag{2.2}$$

where $N(\Sigma)$ is the transversal tubular neighbourhood on the surface Σ.

Using the Abelian dominance approximation and defining the expansion parameter k in units of the bare color charge g with a suitable normalization, $k = 2\pi qg$, the 't Hooft loop operator $M(C)$ becomes

$$M(C) = \mathrm{Tr}\{O_{ij}(C)\} \quad , \tag{2.3}$$

$$O_{ij}(C) = \delta_{ij} \exp\{i2\pi q \int_{M^4} \omega_\Sigma \wedge \beta_j [\cos(g \oint_C \alpha_j) + i \sin(g \oint_C \alpha_j)]\} \quad ,$$

where the configurations α_i are to be intended as singular ones, related to the singularities occurring if two consecutive eigenvalues of B coincide [6]. If $\beta_i = \beta_{i+1}$, we shall label such a point by $x^{(i)}$ and q_i will be the corresponding magnetic charge. The key point will be the identification of the arbitrary expansion parameter q with the magnetic charges q_i, i.e. we will set $q_i \propto q \propto \frac{1}{g}$. Consider then the magnetic order parameter M in the $q \to 0$ limit, corresponding therefore at the strong coupling limit $g \to \infty$. In this limit, and relaxing the constraint $\sum_i \alpha_i = 0$ thus extending the maximal torus of $SU(N)$ to $U(1)^N$ [18], the operator (2.3) can be approximated by

$$M(C) = \sum_i O_{ii}(C) \simeq N \exp\left\{\frac{i2\pi q}{N} \int \omega_\Sigma \wedge \sum_i \beta_i \cos(g \oint_C \alpha_i)\right\} \quad . \tag{2.4}$$

With all these approximations taken into account the *vev* of the magnetic order parameter in the strong coupling region becomes

$$\begin{aligned} < M(C) >_{conn} \simeq & \frac{N}{< 1 >} \int \mathcal{D}\alpha_i \mathcal{D}\beta_i \exp\{-\frac{1}{2} \sum_i \int (\frac{1}{4}\beta_i \wedge *\beta_i + i\beta_i \wedge [d\alpha_i + \frac{4\pi q}{N}\omega_\Sigma \cdot \\ & \cdot \cos(g \oint_C \alpha_i)])\} = \frac{N}{< 1 >} \int \mathcal{D}\alpha_i \exp\{-\frac{1}{4} \sum_i \int [d\alpha_i + \frac{4\pi q}{N}\omega_\Sigma \cos(g \oint_C \alpha_i)]^2)\} \ , \end{aligned} \tag{2.5}$$

where the square of a form t means $t \wedge *t$. We now split the gauge fields as $\alpha_i = \bar{\alpha}_i + \hbar Q_i$, where the quantum fluctuations Q_i must be gauged (e.g. by a covariant gauge condition) and the $\bar{\alpha}_i$ are singular classical configurations. Postponing for a while the discussion of quantum fluctuations, we concentrate on the semi-classical contribution to the path integral which is given by

$$< M(C) >_{conn} \simeq N \exp\{-\frac{8\pi^2 q^2}{N^2} \sum_i \int \omega_\Sigma \wedge \omega_{\Sigma'} \cos(g \oint_C \bar{\alpha}_i) \cos(g \oint_{C'} \bar{\alpha}'_i)\} \quad , \tag{2.6}$$

where Σ' represents a point splitting regularizations of Σ for any point of Σ. The above equation has been obtained observing that the configurations $\bar{\alpha}_i$ obey the (monopole) equation

$$* \, d\bar{\alpha}_i = \frac{4\pi q}{N} \omega_\Sigma \cos(g \oint_C \bar{\alpha}_i) \quad , \tag{2.7}$$

derived using the self-duality and closedness of ω_Σ. Due to the singular behaviour of the $\bar{\alpha}_i$'s partial integration is not allowed [19] and no electric currents are present in the model. Equations of the type (2.7) appeared already in the study of the duality properties of gauge theories and 4D manifold invariants [20, 21].

The classical contribution to the action is

$$S_0 = \frac{2\pi^2 m^2}{g^2} \int \sum_i \omega_\Sigma \wedge \omega_{\Sigma'} \cos(g \oint_C \bar{\alpha}_i) \cos(g \oint_{C'} \bar{\alpha}'_i) \equiv \frac{2\pi^2 m^2 N}{g^2} Q(\Sigma, \Sigma') \quad , \tag{2.8}$$

where Dirac quantization $g \oint_C \bar{\alpha}_i = 2\pi n$ has been used, $m \in \mathbb{Z}$ labels monopole charges and $Q(\Sigma, \Sigma')$ is the algebraic intersection number [22]. The closed curve $C' \equiv \{y^\mu(t)\}$ corresponds to the framing contour of $C \equiv \{x^\mu(t)\}$, i.e. if it happens that $y^\mu(t) = x^\mu(t) + \epsilon n^\mu(t)$, with $\epsilon \to 0$, and $|n^\mu| = 1$ where $n^\mu(t)$ is a vector field orthogonal to

C. Then $Q(\Sigma,\Sigma')$ becomes the self-linking number of C, $Q(\Sigma,\Sigma') = \mathrm{sLink}(C)$. From a physical point of view, we may define sLink(C) as

$$\mathrm{sLink}(C) \equiv \frac{L(C)}{\rho} \quad , \tag{2.9}$$

where $L(C)$ is the perimeter of the loop C and ρ plays the role of unit of lenght, reasonably associated to the vortex penetration depth.

Let us now discuss quantum fluctuations. If the effective theory for large distances is a $U(1)$ type theory, for short distances the charged degrees of freedom cannot be discarded anymore. Let Λ be the scale separating these two regimes and let us divide the gauge field accordingly; moreover let, for the sake of simplicity, the gauge group be $SU(2)$. For scales bigger than Λ we take $A^3 = \bar{\alpha} + Q^3$ which is the usual $U(1)$ prescription. For scales smaller than Λ we take $A^a = \delta^{a3}\bar{\alpha} + Q^a$, *i.e.* we continue the classical solution into the small scales region where the quantum fluctuations coming from the charged degrees of freedom cannot be discarded. The expectation is that the small scales behaviour is insensitive to the classical solution according to the background field method. Performing the functional integration over the quantum fluctuations leads to a double contribution, in complete analogy with the saddle point evaluation around an instanton background [23]. The first contribution is given by a ratio of determinants given by

$$R = \left[\frac{Det'(-L_0)}{Det'(-L)}\right]^{\frac{1}{2}} \left[\frac{Det(-\bar{D}^2)}{Det(-\partial^2)}\right] \quad , \tag{2.10}$$

where $\bar{D} = d - ig\bar{\alpha}$, $\bar{D}^2 \equiv \bar{D}^\mu \bar{D}_\mu$, $L = \bar{D}^2\delta_{\mu\nu} - (1-\frac{1}{\xi})\bar{D}_\mu\bar{D}_\nu + 2ig\bar{F}^0_{\mu\nu}(x)$, and L_0 is given from L evaluated around the trivial background. ξ is the gauge parameter and in the primed determinants zero modes are omitted.

The second contribution is given by the Pauli-Villars regularization of the determinants and it amounts to a scale μ (which is the Pauli-Villars mass) raised to a certain power which is given by the dimension of the moduli space of the classical solution.

Let us now proceed with the evaluation of these two contributions. Using the self-duality property of our classical solution (2.7), the ratio of determinants (2.10) can be written as $R = \left[Det(-\partial^2)/Det(-\bar{D}^2)\right]$ [23]. This ratio has been evaluated in Ref.[24] using the heat kernel method in the case of an $SU(N)$ gauge group but it is easy to specialize this result to our case. We obtain

$$\ln \frac{Det(-\partial^2)}{Det(-\bar{D}^2)} = \frac{\ln(\mu\Lambda)}{96\pi^2}\int \bar{F}^0_{\mu\nu}(x)\bar{F}^0_{\mu\nu}(x)d^4x = \frac{\ln(\mu\Lambda)}{96\pi^2}\frac{1}{4}\int (d\bar{\alpha})_{\mu\nu}(x)(d\bar{\alpha})_{\mu\nu}(x)d^4x \quad . \tag{2.11}$$

The factor 1/4 comes from the normalization of the gauge group generators according to the reduced connections $\bar{\alpha}_i$.

The contribution coming from the regularization of the zero modes is obtained once the dimension of the moduli space is computed, according to Ref.[21], to be

$$\dim \mathcal{M} = c_1(L)^2 = c_1(L)\wedge c_1(L) = \frac{1}{32\pi^2}\int (d\bar{\alpha})_{\mu\nu}(x)(d\bar{\alpha})_{\mu\nu}(x)d^4x \quad . \tag{2.12}$$

Putting together the classical result (2.8) with the quantum fluctuations, we find that the bare coupling g can be substituted by its renormalized expression and that

the exponent in (2.6) can be written as

$$\frac{8\pi^2}{g_R^2}\frac{c_1(L)^2}{8} = \frac{c_1(L)^2}{8}\left(\frac{8\pi^2}{g^2} - (8-\frac{2}{3})\ln(\mu\Lambda)\right) \equiv \frac{c_1(L)^2}{8}\left(\frac{8\pi^2}{g^2} - \beta_1\ln(\mu\Lambda)\right) \quad , \tag{2.13}$$

where $\beta_1 = \frac{22}{3}$ is the first coefficient of the $SU(2)$ beta function of the non-abelian Yang-Mills theory.

THE AVERAGE OF THE WILSON LOOP

In this section we shall compute the average of the Wilson loop and find an area law behaviour for its leading part. Furthermore, in our formalism, the area law gets a nice geometrical interpretation: it is the response of the true QCD vacuum to arbitrary deformations of the quark loop $\mathcal{C}$.

The starting point here is given by the Wilson loop operator written in terms of the non abelian Stokes theorem (see e.g. [13]):

$$W_t(\mathcal{C}) \equiv W_t(\Sigma, C) = Tr_t\{P_\Sigma \exp[i\int_\Sigma \mathrm{Hol}_{\bar{x}}^{x}(\gamma)F(x)\mathrm{Hol}_{x}^{\bar{x}}(\gamma')]\} \quad , \tag{3.1}$$

where $\mathcal{C} = \partial\Sigma$, $C = \{\gamma(x) \cup \gamma'(x)\}$ was defined at the beginning of section 2 and P_Σ means surface path ordering. W_t is calculated with respect to some irreducible representation t of $SU(N)$.

$$< W_t(\mathcal{C}) >_{conn} \equiv \frac{< W_t(\Sigma, C) >}{< 1 >} \equiv \frac{\int \mathcal{D}B\mathcal{D}A\ W_t(\Sigma, C)e^{-S_{BF-YM}}}{\int \mathcal{D}B\mathcal{D}A\ e^{-S_{BF-YM}}} \quad , \tag{3.2}$$

where S_{BF-YM} was defined in (1.1). Expanding in series the Wilson loop (3.1) we get

$$< W_t(\Sigma, \mathcal{C}) >= \sum_n < \frac{1}{n!}\mathrm{Tr}_t P_\Sigma \int_{\Sigma_1} \cdots \int_{\Sigma_n} (i\mathrm{Hol}(\gamma)F(x)\mathrm{Hol}^{-1}(\gamma'))^n > \quad . \tag{3.3}$$

We then use the identity

$$e^{-\frac{i}{4}\int *B^a_{\mu\nu}F^a_{\mu\nu}}F^a_{\rho\sigma}(x) = 4i\frac{\delta}{\delta * B^a_{\rho\sigma}(x)}(e^{-\frac{i}{4}\int *B^a_{\mu\nu}F^a_{\mu\nu}}) \quad , \tag{3.4}$$

and performing a partial integration with respect to the functional derivative in (3.4) we can replace, in the path integral,

$$i\mathrm{Hol}(\gamma)e^{-\frac{i}{4}\int *B^a\cdot F^a}F(x)\mathrm{Hol}^{-1}(\gamma') \to 4\mathrm{Hol}(\gamma)e^{-\frac{i}{4}\int *B^a\cdot F^a}(\frac{\delta}{\delta * B(x)})\mathrm{Hol}^{-1}(\gamma') \quad . \tag{3.5}$$

The functional derivative acts only to its right on the exponential of the mass term $-g^2/16\int B^a_{\mu\nu}B^a_{\mu\nu}$, since Hol does not contain the B field. We now need the identity

$$V(\frac{\delta}{\delta * B^a(x)_{\rho\sigma}})e^{-\frac{g^2}{16}\int B^a_{\mu\nu}B^a_{\mu\nu}} = V(-\frac{g^2}{8} * B^a_{\rho\sigma})e^{-\frac{g^2}{16}\int B^a_{\mu\nu}B^a_{\mu\nu}} \quad , \tag{3.6}$$

where V is the functional defined by

$$V(\frac{\delta}{\delta * B^a(x)}) \equiv P_\Sigma \int_{\Sigma_1} \cdots \int_{\Sigma_n} (4\mathrm{Hol}(\gamma)(\frac{\delta}{\delta * B^a(x)})\mathrm{Hol}^{-1}(\gamma'))^n \quad . \tag{3.7}$$

Resumming the exponential series for the Wilson loop we finally get the "duality" relation

$$< W_t(\mathcal{C}) >_{conn} = \frac{< M_t^*(\Sigma, C, \mathcal{C} = \partial\Sigma) >}{< 1 >} \quad , \tag{3.8}$$

where

$$M_t^*(\Sigma, C, \mathcal{C} = \partial\Sigma) = Tr_t[P_\Sigma \exp\{-\frac{g^2}{2}\int_\Sigma \mathrm{Hol}_{\bar{x}}^x(\gamma) * B(x)\mathrm{Hol}_x^{\bar{x}}(\gamma')\}] \quad . \tag{3.9}$$

$M_t^*(\Sigma, C)$ is the dual (in the sense that $B \to *B$) of the observable $M_t(\Sigma, C)$ defined in (1.3) with k set to $k = ig^2/2$.

To calculate (3.8) we expand perturbatively in g both the exponential and the holonomies which appear in the exponent of (3.9) [25]. The first relevant contraction encountered at lower level is given in terms of $< A * B >$, which can be computed starting from the off-diagonal propagator $< AB >$ [12]. Therefore we find

$$< M_t^*(\mathcal{C}) >_{conn} = e^{-\frac{g^2}{2}c_2(t)\oint_C\int_\Sigma < A*B >}\Delta(\Sigma) \quad . \tag{3.10}$$

$\Delta(\Sigma)$ depends on higher order integrations over Σ which do not simply involve the quantity $\oint_C\int_\Sigma < A * B >$ alone; therefore while the explicit calculation of $\Delta(\Sigma)$ is an open problem, for the purpose of showing the area law behaviour its knowledge should not be essential. Consider now the integral in the exponent of (3.10),

$$\begin{aligned}
&\oint_C\int_\Sigma < A * B > = < \oint_C A\int_\Sigma d\sigma^{\mu\nu}(x)(*B(x))_{\mu\nu} > \qquad (3.11)\\
&= < \oint_C A\int_\Sigma (*d\sigma)^{\mu\nu}(x)B(x)_{\mu\nu} > = \oint_C dy^\lambda\int_\Sigma (*d\sigma)^{\mu\nu}(x) < A(y)_\lambda B(x))_{\mu\nu} > ,
\end{aligned}$$

where $*d\sigma(x)$ is the infinitesimal surface element of the plane Σ_x^* dual to Σ at the point $x \in \Sigma$. We may rewrite

$$\oint_C dy^\lambda(*d\sigma)^{\mu\nu}(x) < A(y)_\lambda B(x))_{\mu\nu} > = \oint_C dy^\lambda\int_{\Sigma_x^*}\omega_\Sigma^{\mu\nu} < A_\lambda B_{\mu\nu} > \quad , \tag{3.12}$$

recalling that ω_Σ is locally given by a bump function with support on Σ. Eq. (3.12) is by definition the linking number between the curve C and the dual plan Σ_x^* in x to Σ. Indeed the linking $\mathrm{Link}(C, \Sigma)$, whith arbitrary C and Σ, is defined by [26]

$$\begin{aligned}
\mathrm{Link}(C, \Sigma) &= \frac{1}{8\pi^2}\oint_C dx^\alpha\int_\Sigma d\sigma^{\beta\gamma}(y)\epsilon_{\alpha\beta\gamma\delta}\frac{(x-y)^\delta}{|x-y|^4}\\
&= 4\oint_C dx^\alpha\int_\Sigma d\sigma^{\beta\gamma}(y) < A_\alpha^a(x)B_{\beta\gamma}^a(y) > \quad .
\end{aligned} \tag{3.13}$$

In our case, by construction, $\mathrm{Link}(C, \Sigma_x^*) \neq 0$. The residual integration over Σ in (3.12) spans all the dual Σ^* to Σ, yielding a contribution proportional to the area of Σ,

$$\begin{aligned}
< W_t(C) >_{conn} &\sim \exp\{-\frac{g^2c_2(t)}{8}\int_\Sigma \mathrm{Link}(C, \Sigma_x^*)\} \quad ,\\
\int_\Sigma \mathrm{Link}(C, \Sigma_x^*) &\sim A(\Sigma) \quad .
\end{aligned} \tag{3.14}$$

We may get a better understanding of (3.14) considering a lattice regularization of Σ, i.e. $\Sigma \to \Sigma_{PL}$. In this case C runs over the links of the lattice, while Σ_x^* corresponds

to an element of the dual lattice through the plaquette centered at x. $\int_{\Sigma_{PL}} \text{Link}(C, \Sigma_x^*)$ is an integer which counts the number N_v of vertices of the dual lattice on Σ_{PL} or equivalently the number of plaquettes N_P of Σ_{PL}. We may then write

$$\int_{\Sigma_{PL}} \text{Link}(C, \Sigma_x^*) = N_P = \frac{A(\Sigma_{PL})}{a^2} \quad , \tag{3.15}$$

where $A(\Sigma_{PL})$ is the minimal area bounded by the quark loop $\mathcal{C}$ and a is the lattice spacing. Up to this point the quantities which enter the calculations are the bare ones, but when passing to the continuum limit $a \to 0$ the renormalization of the theory implies that they are replaced by the renormalized ones. In particular $N_P = A(\Sigma_{PL})/a^2$ should become $A(\Sigma)/l^2$ with l a typical scale in QCD which, owing to a proper dressing of the off diagonal propagator entering (3.10), should be choosen as Λ_{QCD}^{-1}. Therefore one may rewrite (3.14) as

$$< W_t(\mathcal{C}) >_{conn} \sim \exp[-\sigma(\Lambda_{QCD}) A(\Sigma)] \quad , \tag{3.16}$$

where $\sigma(\Lambda_{QCD})$ is the string tension defined by

$$\sigma(\Lambda_{QCD}) \equiv \frac{g_R^2 c_2(t)}{8} \Lambda_{QCD}^2 \tag{3.17}$$

and where we have replaced the bare coupling constant g with the running coupling constant g_R. The way in which the continuum limit is reached and the fields are dressed is presently not under our control but the key geometrical feature of the area law should remain unchanged.

A deeper investigation should also clarify some other questions. While the area law displayed in (3.16) corresponds to the expected confining behaviour of QCD vacuum, no screening is shown when the test charges are taken in the adjoint representation and the behaviour in this case is closer to what expected in the large N limit [27]. It might seem puzzling that the area law behaviour comes out of a perturbative calculation. In reality our perturbative expansion has to be understood as an expansion around the gauge fixed topological BF theory whose vacuum contains "relevant" topologically non-trivial configurations. It is thus resonable to imagine that the non perturbative information comes out of this non-trivial vacuum structure. The contribution of these "magnetically charged" configurations is expected to give rise to a mass gap in the model and we guess a possible form for it, namely

$$m^2 \sim \mu^2 e^{-\frac{2\pi}{q^2}} \quad , \tag{3.18}$$

where μ is a regulator mass and q is the magnetic charge satisfying a proper Dirac quantization $qg = 2\pi n$. A suitable dressing displaying such a pole for propagators is obtained substituting (in euclidean momentum space) the $1/p^2$ denominators with $1/(p^2 + \Pi(p^2))$, where $\Pi(p^2) \sim 2(\frac{q^2}{4\pi})p^2 \ln(\frac{p^2}{\mu^2})$. In such a way, in a $q \to 0$ limit (i.e. strong coupling expansion) and applying Dirac quantization, the string tension read out of (3.14) should be vanishing when the adjoint representation is choosen.

A last point to be discussed is the apparent surface dependence which the Wilson loop acquires in the representation (3.8). Deforming the surface bordered by the loop $\mathcal{C}$ corresponds to a "gauge" transformation on the connections defined on the loop space; in this sense the "gauge" independence of observables in loop space corresponds to

surface independence for the Wilson line. Lastly, our formalism appears quite similar to that recently introduced in [28] for the *vev* of W.

M.M. acknowledges the hospitality of the Board of the Cargese School. This work has been partially supported by MURST. M. Z. is associated to the University of Milan according to TMR programme ERB-4061-PL-95-0789.

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